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Split and Strong Split Steiner Domination Number of Graphs

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Abstract:-- **In this paper, split and strong split steiner domination number of a graph are introduced. Also, these numbers were found for some standard graphs.**

Keywords:-- **steiner number, steiner domination number, split steiner domination number, strong split steiner domination number.**

1. INTRODUCTION

The concept of domination in graphs was introduced by Ore and Berge [4]. Throughout this paper G = (V, E) denotes a finite undirected simple graph with vertex set V and edge set E. A subset D of V(G) is a dominating set of G if every vertex in V − D is adjacent to at least one vertex in D. The minimum cardinality of a dominating set of G is called the domination number of G and is denoted by γ(G). The concept of Steiner number of a graph was introduced by G. Chatrand and P. Zhang [1]. For a nonempty set W of vertices in a connected graph G, the Steiner distance d(W) of W is the minimum size of a connected subgraph of G containing W . Necessarily each such subgraph is a tree and is called a Steiner tree with respect to W or a Steiner W-tree. The set of all vertices of G that lie in some Steiner W-tree is denoted by S(W). If S(W) = V, then W is called a Steiner set for G. A Steiner set with minimum cardinality is the Steiner number of G and is denoted by s(G).

The concept of Steiner domination number of a graph was introduced by J. John et al., [3]. For a connected graph G, a set of vertices W in G is called a Steiner dominating set if W is both a Steiner set and a dominating set. The minimum cardinality of a Steiner dominating set of G is its Steiner domination number and is denoted by $\gamma_s(G)$. A steiner dominating set of cardinality $\gamma_s(G)$ is said to be a γ_s-set.

A dominating set S of a graph G is said to be split dominating set of G if the subgraph induced by V − S is disconnected. The minimum cardinality among all split dominating sets is called the split domination number of G. A dominating set S of a graph G is said to be strong split dominating set of G if the subgraph induced by V − S is totally disconnected or independent. The minimum cardinality among all strong split dominating sets is called the strong split domination number of G and is denoted by $\gamma_{ss}(G)$.

A vertex and an edge are said to cover each other if they are incident. A set of vertices which covers all the edges of a graph G is called a vertex cover for G. The smallest number of vertices in any vertex cover for G is called its vertex covering number and is denoted by $\alpha_0(G)$ or α_0. A set S of vertices in a graph G is independent if no two of its vertices are adjacent in G. The largest number of vertices in such a set is called the vertex independence number of G and is denoted by $\beta_0(G)$ or β_0. If G is a graph with p vertices, then $\alpha_0(G) + \beta_0(G) = p$.

Theorem 1.1.[3] *For the complete bipartite graph* $G = K_{m,n}$,

$$s(G) = \gamma_s(G) = \begin{cases} 2 & \text{if } m = n = 1 \\ n & \text{if } n \geq 2, m = 1 \\ \min\{m,n\} & \text{if } m,n \geq 2 \end{cases}$$

Theorem 1.2.[6] *For a Wheel graph* $W_{1,n}, n \geq 5, \gamma_s(W_{1,n}) = n-2$.

Theorem 1.3.[3] *For the complete graph* $K_p (p \geq 2), \gamma_s(K_p) = p.$

Theorem1.4.[5] $\gamma_s(P_n) = \begin{cases} \left\lceil \frac{n-4}{3} \right\rceil + 2 & \text{if } n \geq 5; \\ 2 & \text{if } n = 2,3 \text{ or } 4. \end{cases}$

Theorem 1.5.[5] *For n > 5,* $\gamma_s(C_n) = \left\lceil \frac{n}{3} \right\rceil$.

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2. SPLIT STEINER DOMINATION NUMBER

Definition 2.1. A steiner dominating set W of G is said to be a split steiner dominating set of G if the sub graph induced by V − W is disconnected.

Definition 2.2. Let ζ' denote the collection of all graphs having atleast one split steiner dominating set. Let $G \in \zeta'$. Then, the minimum cardinality of all split steiner dominating sets of G is called the split steiner domination number of G. It is denoted by $\gamma_s^s(G)$. A split steiner dominating set of cardinality $\gamma_s^s(G)$ is called a γ_s^s-set of G.

Example 2.3. Consider the graph G in figure 2.1. Here, W = $\{v_1, v_4, v_6, v_7\}$ is a minimum steiner dominating set of G which is also a minimum split steiner dominating set of G. Therefore, $\gamma_s^s(G)=\gamma_s(G)=4$.

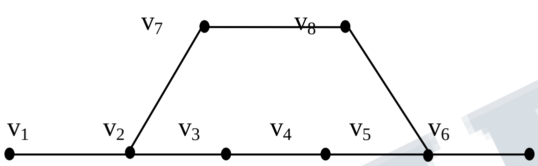

Figure 2.1

Example 2.4. Consider the graph G in figure 2.2. Here, W = $\{v_1, v_4\}$ is the minimum steiner dominating set of G, and so $\gamma_s(G) = 2$. But, W is not a split steiner dominating set of G. Here $W_1 = \{v_1, v_2, v_4\}$ is the minimum split steiner dominating set of G. Therefore, $\gamma_s^s(G) = 3$.

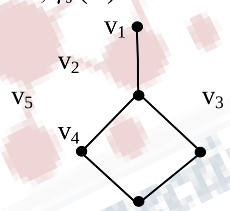

Figure 2.2

Observation 2.5. The following are observed for any connected graph G.

i. A complete graph has no split steiner dominating set, as the vertex set is the unique steiner dominating set of it.

ii. In general, all graphs need not have split steiner dominating sets.

Observation 2.6. Let $G \in \zeta'$. Then, the following are observed.

i. Every split steiner dominating set is a steiner dominating set of G. Therefore, $\gamma_s^s(G) \geq \gamma_s(G)$.

ii. Every split steiner dominating set is a split dominating set of G. Hence, split steiner domination number of G is greater than or equal to split domination number.

iii. Every extreme vertex of G belongs to every split steiner dominating set of G.

iv. $2 \leq \gamma_s(G) \leq \gamma_s^s(G) \leq p$.

Proposition 2.7. For $n \geq 5$, $\gamma_s^s(P_n) = \gamma_s(P_n)$.

Proof. Let $n \geq 5$. Clearly, for every minimum steiner dominating set W of P_n, V−W is disconnected. Therefore, every steiner dominating set of P_n is a split steiner dominating set of it and so $\gamma_s^s(P_n) \leq \gamma_s(P_n)$. By Observation 2.6 (iv), $\gamma_s^s(P_n) = \gamma_s(P_n)$.

Proposition 2.8. For $n > 3$, $\gamma_s^s(C_n) = \gamma_s(C_n)$.

Proof. Let $n > 3$. Clearly, for every minimum steiner dominating set W of C_n, V−W is disconnected. Therefore, every steiner dominating set of C_n is a split steiner dominating set of it and so $\gamma_s^s(C_n) \leq \gamma_s(C_n)$. By Observation 2.6(iv), $\gamma_s^s(C_n) = \gamma_s(C_n)$.

Corollary 2.9. By Theorem 1.5, For $n > 3$, $\gamma_s^s(C_n) = \gamma_s(C_n) = \left\lceil \frac{n}{3} \right\rceil$.

Proposition 2.10. The Wheel graph $W_{1,p}$, $p \geq 5$ has no split steiner dominating set.

Proof: Let W be the minimum steiner dominating set of $W_{1,p}$. Then, by Theorem 1.2, $|W| = p - 2$. Obviously, V – W is connected and hence W is not a split steiner dominating set of $W_{1,p}$. Since, W is arbitrary the wheel graph $W_{1,p}$, $p \geq 5$ has no split steiner dominating set.

Proposition 2.11. Let $m, n \geq 2$. Then, $\gamma_s^s(K_{m,n}) = \min\{m, n\}$.

Proof. Let S and T be the bipartitions of $K_{m,n}$ with $|S| = m$ and $|T| = n$. Let W be a minimum steiner dominating set of $K_{m,n}$. Then, by Theorem 1.1, W = S or T. Therefore, V – W = T or S which is disconnected. Hence, W is a split steiner

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dominating set of G. Therefore, $\gamma_s^s(K_{m,n}) = \gamma_s(K_{m,n}) = \min\{m, n\}$.

Proposition 2.12. Let G be a connected graph on p vertices. Then, if $G \in \zeta'$ then $G^+ \in \zeta'$ and $\gamma_s^s(G^+) \le p + \gamma_s^s(G)$.

Proof. Let V (G) = $\{v_1, v_2, ..., v_p\}$ and $w_1, w_2, ..., w_p$ be the end vertices attached to $v_1, v_2, ..., v_p$ respectively in G^+. If D is a minimum split steiner dominating set of G, then W = $\{w_1, w_2, ..., w_p\} \cup D$ is a split steiner dominating set of G^+ and so $\gamma_s^s(G^+) \le p + \gamma_s^s(G)$.

3. STRONG SPLIT STEINER DOMINATION NUMBER

Definition 3.1. A steiner dominating set W of G is said to be a strong split steiner dominating set of G if the subgraph induced by V − W is totally disconnected. That is, the subgraph induced by V − W is independent.

Definition 3.2. Let ζ'' denote the collection of all graphs having atleast one strong split steiner dominating set. Let $G \in \zeta''$. Then, the minimum cardinality of all strong split steiner dominating sets of G is called the strong split steiner domination number of G. It is denoted by $\gamma_s^{ss}(G)$. A strong split steiner dominating set of cardinality $\gamma_s^{ss}(G)$ is called a γ_s^{ss}-set of G.

Example 3.3. Consider the graph G in figure 3.1. Here, W = $\{v_1, v_3\}$ is a minimum steiner dominating set of G and is also a minimum strong split steiner dominating set of G. Therefore, $\gamma_s^{ss}(G) = \gamma_s(G) = 2$.

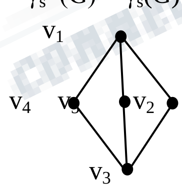

Figure 3.1

Example 3.4. Consider the graph G in figure 3.2. Here, W = $\{v_1, v_4, v_5\}$ is the minimum steiner dominating set of G, and so $\gamma_s(G) = 3$. But, W is not a strong split steiner dominating set of G. Here, $W_1 = \{v_1, v_2, v_4, v_5\}$ is the minimum strong split steiner dominating set of G. Therefore, $\gamma_s^{ss}(G) = 4$.

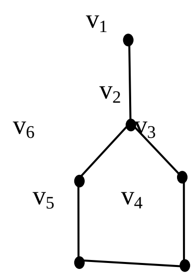

Figure 3.2

Observation 3.5. The following are observed for any connected graph G.

i. A complete graph has no strong split steiner dominating set, as the vertex set is the unique steiner dominating set of it.

ii. In general, all graphs need not have strong split steiner dominating sets.

Observation 3.6. Let $G \in \zeta''$. The following are observed.

i. Every strong split steiner dominating set is a split steiner dominating set of G and further a steiner dominating set of G. Therefore, $\gamma_s^{ss}(G) \ge \gamma_s^s(G) \ge \gamma_s(G)$.

ii. Every strong split steiner dominating set is a strong split dominating set of G and further a split dominating set of G. Therefore, $\gamma_s^{ss}(G) \ge \gamma_{ss}(G) \ge$ split domination number of G.

iii. Every extreme vertex of G belongs to every strong split steiner dominating set of G.

iv. $2 \le \gamma_s(G) \le \gamma_s^s(G) \le \gamma_s^{ss}(G) \le p$.

Proposition 3.7. Let $G \in \zeta''$. Then, $\gamma_s^{ss}(G) \ge \alpha_0(G)$.

Proof: Let W be a strong split steiner dominating set of G. Then V − W is independent. Therefore, $|V-W| \le \beta_0(G)$. That is, $p - \gamma_s^{ss}(G) \le \beta_0(G) = p - \alpha_0(G)$. Hence, $\gamma_s^{ss}(G) \ge \alpha_0(G)$.

Proposition 3.8. For n > 3, $\gamma_s^{ss}(P_n) = \begin{cases} \left\lceil \frac{n}{2} \right\rceil & \text{if } n \text{ is odd} \\ \frac{n}{2}+1 & \text{if } n \text{ is even} \end{cases}$

Proof. Let n > 3 and let $P_n = (v_1, v_2, ..., v_n)$.

Case 1: n is odd.

Let $W = \{v_1, v_3, ..., v_n\}$. Obviously, W is a strong split steiner dominating set of P_n. Since any two consecutive vertices are adjacent in P_n, W is a minimum strong split steiner dominating set of P_n. Hence, $\gamma_s^{ss}(P_n) = |W| = \left\lceil \frac{n}{2} \right\rceil$ *if n is odd*.

Case 2: n is even.

Let $W=\{v_1,v_3,...,v_{n-1},v_n\}$. Proceeding as in case (i), $\gamma_s^{ss}(P_n) = |W| = \frac{n}{2}+1$.

Proposition 3.9. For n > 3, $\gamma_s^{ss}(C_n) = \left\lceil \frac{n}{2} \right\rceil$.

Proof. Let n > 3 and let $C_n = (v_1, v_2, ..., v_n, v_1)$.

Case 1: n is odd.

Clearly, $W=\{v_1,v_3,...,v_n\}$ is a strong split steiner dominating set of C_n.

Therefore, $\gamma_s^{ss}(C_n) \le |W| = \left\lceil \frac{n}{2} \right\rceil$ --------(1)

Further, if W is a strong split steiner dominating set of C_n, then V – W is independent and so maximum cardinality of V– W is $\left\lfloor \frac{n}{2} \right\rfloor$.

Therefore, $|V - W| \le \left\lfloor \frac{n}{2} \right\rfloor$. That is, $|W| \ge |V| - \left\lfloor \frac{n}{2} \right\rfloor = \left\lceil \frac{n}{2} \right\rceil$ as n is odd.

Therefore, $\gamma_s^{ss}(C_n) \ge \left\lceil \frac{n}{2} \right\rceil$ -------- (2)

From (1) and (2), it is proved that if n > 3 is odd, then $\gamma_s^{ss}(C_n) = \left\lceil \frac{n}{2} \right\rceil$.

Case 2: n is even.

Clearly, $W = \{v_1,v_3,...,v_{n-1}\}$ is a strong split steiner dominating set of C_n and so $\gamma_s^{ss}(C_n) \le |W| = \frac{n}{2}$ --------(3)

Further, if S is a strong split steiner dominating set of C_n, then V – W is independent and so maximum cardinality of V – W is $\frac{n}{2}$. Therefore, $|V - W| \le \frac{n}{2}$. That is, $|W| \ge |V| - \frac{n}{2}$.

That is, $|W| \ge \frac{n}{2}$ as n is even. Therefore, $\gamma_s^{ss}(C_n) \ge \frac{n}{2}$ ----- (4)

In other words, from (3) and (4), $\gamma_s^{ss}(C_n) = \frac{n}{2}$ if n is even.

Hence, in general, for n>3, $\gamma_s^{ss}(C_n) = \left\lceil \frac{n}{2} \right\rceil$.

Proposition 3.10. The Wheel graph $W_{1,p}$, p ≥5, has no strong split steiner dominating set.

Proof: Let W be the minimum steiner dominating set of $W_{1,p}$. Then, by Theorem 1.2, |W|= p – 2 . Obviously, V – W is connected and hence W is not a strong split steiner dominating set of $W_{1,p}$. Since, W is arbitrary the wheel graph $W_{1,p}$, p ≥5 has no strong split steiner dominating set.

Proposition 3.11. For m, n ≥ 2, $\gamma_s^{ss}(K_{m,n}) = \min\{m, n\}$.

Proof. Let S and T be the bipartitions of $K_{m,n}$ with |S| = m and |T| = n. Let W be a minimum steiner dominating set of $K_{m,n}$. Then, by Theorem 1.1, W = S or T. Therefore, V – W = T or S which is totally disconnected. Hence, W is a strong split steiner dominating set of G. Therefore, $\gamma_s^{s}(K_{m,n}) = \gamma_s(K_{m,n}) = \min\{m, n\}$.

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