A Note on SG – algebras

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Abstract:-- In this paper we introduce the notion of SG –algebras as a generalization of BCK/BCI/Q/TM algebras and investigated some of the elementary properties by comparing with other algebras.

Keywords:--BCI algebra, BCH algebra, TM algebras, SG algebras.

1. INTRODUCTION

Imai and Iseki (1966) introduced a class of abstract algebra known as BCK – algebra. At the same time, Iseki introduced another class of algebra, called BCI – algebra and investigated some of its properties. It is known that class of BCK – algebra is a proper sub class of the class of BCI – algebra. These two are the important classes logical algebras. Hu and Li (1985) introduced a wide class of abstract algebras. Also we define the binary operation A –algebra unless otherwise specified. In X we define a binary relation ≤ by x ≤ y if and only if x * y = 0. Note that every QS – algebra is a Q – algebra. A BCI – algebra is an algebra (X,*,0) of type (2,0) satisfying the following conditions: i) (x * y) * (x * z) ≤ z * y

2. Preliminaries

A Q – algebra is a non – empty set X with a constant 0 but a binary operation * satisfying the axioms:

i) x * x = 0

ii) x * 0 = x

iii) (x * y) * z = (x * z) * y for all x, y, z ∈ X.

A Q – algebra (X,*,0) is called a QS – algebra if (x * y) * (x * z) = z * y for all x, y, z ∈ X.

For brevity we shall call X a QS – algebra unless otherwise specified. In X we define a binary relation ≤ by x ≤ y if and only if x * y = 0. Note that every QS – algebra is a Q – algebra. A BCI – algebra is an algebra (X,*,0) of type (2,0) satisfying the following conditions:

i) (x * y) * (x * z) ≤ z * y

Theorem 1.1

Definition 3.1

A SG – algebra (X,*,0) is a non – empty set X with a constant 0 and a binary operation * satisfying the axioms:

i) x * 0 = x for x ∈ X

ii) x * (x * y) = y for all x, y ∈ X

In X we define a binary relation ≤ by x ≤ y if and only if x * y = 0

Example 3.2

Let Z be the set of all integers and let nZ = {nx: x ∈ Z}, n ∈ Z. Then (Z,+,0) and (nZ,−,0) are SG – algebras. (where ‘−’ is the usual subtraction)
Solution:

i) \( x - 0 = x \) for all \( x \in Z \)

ii) \( x - (x - y) = x - x + y = y \) for all \( x, y \in Z \)

Hence \((Z, -, 0)\) is a SG – algebra.

i) \( nx = nx \) for all \( n, x \in Z \)

ii) \( nx - (nx - ny) = nx - nx + ny = ny \) for all \( n, x, y \in Z \)

Hence \((nZ, -, 0)\) is a SG – algebra.

Example 3.3
Let \( X = \{0, 1, 2, 3\} \) be a set with cayley table

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Then \((X, *, 0)\) is a SG – algebra.

**Proposition 3.4**
If \((X, *, 0)\) is a SG – algebra, then \( x * x = 0 \).

**Proof**
\( x * x = x * (x * 0) = 0 \).

**Proposition 3.5**
Let \((X, *, 0)\) be a SG – algebra. Then for any \( x, y, z \in X \)

i) \( x * 0 = 0 \) implies \( x = 0 \)

ii) \( x * (x * (x * y)) = x * y \)

iii) \( (x * (x * y)) * y = y * y = 0 \) (by proposition 3.4)

iv) If \( x * y = 0 \) then \( x * x = 0 \), \( x * y = x \) \( x = y \)

**Proof**

i) Given \( x * 0 = 0 \)

\( x = 0 \).

ii) \( x * (x * (x * y)) = x * y \)

iii) \( (x * (x * y)) * y = y * y = 0 \) (by proposition 3.4)

iv) If \( x * y = 0 \) then \( x = x * 0 = x * (x * y) = y \)

Similarly, if \( y * x = 0 \) then \( y = y * 0 = y * x \)

**Theorem 3.6**
Every SG – algebra is a BCI – algebra

**Proof**

Let \((X, *, 0)\) be a SG – algebra.

i) To prove \((x * y) * (x * z) \leq z * y\)

Consider \((x * y) * (x * z)\),

Replace \( y \) by \( x * z \), then \((x * (x * z)) * (x * z) = z * (x * z) = z * y\)

ii) To prove \( x * (x * y) = y \)

This is directly follows from definition

iii) To Prove \( x \leq x \) (ie) to prove \( x * x = 0 \), \( x * x = x * (x * 0) = 0 \)

iv) To Prove \( x \leq y \) and \( y \leq x \) implies \( x = y \)

\( x \leq y \implies x * y = 0 \)

Therefore \( x * (x * y) = x * 0 = x \)

But \( x * (x * y) = y \)

Hence \( y = x \).

Similarly, \( y \leq x \implies y * x = 0 \)

\( y * (y * x) = y * 0 = y \)

But \( y * (y * x) = x \)

Hence \( x = y \).

v) To Prove \( x \leq x \) => \( x * x = 0 \)

\( x \leq 0 \implies x * x = 0 \)

\( x * (x * x) = 0 \implies x = 0 \)

Hence SG – algebra is a generalization of BCI – algebra.

**Theorem 3.7**
Every SG – algebra is a BH – algebra

**Proof**

Let \((X, *, 0)\) be a SG – algebra.

i) \( x * x = 0 \) is directly follows from Proposition 3.4.

ii) \( x * 0 = x \) is directly follows from definition 3.1

iii) \( x * y = 0 \) and \( y * x = 0 \) => \( x = y \) for all \( x, y \in X \), by proposition 3.5(iv).

Hence SG – algebra is a BH – algebra.

**Theorem 3.8**
Every SG – algebra is a BCH – algebra

**Proof**

Let \((X, *, 0)\) be a SG – algebra.

Every BCI – algebra is a BCH – algebra. But SG – algebra is a BCI – algebra

Hence SG – algebra is a BCH - algebra.

**Theorem 3.9**
Every Q – algebra is a SG – algebra

**Proof**

Every SG – algebra is a BCI – algebra

In BCI – algebra, condition (iii) for Q – algebra is satisfied.

Therefore SG – algebra is a Q – algebra.

**Theorem 3.10**
Every SG – algebra is a BG – algebra

**Proof**

Let \((X, *, 0)\) be a SG – algebra.

i) \(x * x = 0\) follows from Proposition 3.4

ii) \(x * 0 = x\) is directly follows from definition.

iii) To prove \(x * (y * z) = (x * y) * z\)

In SG - algebra, \((x * y) * z = (x * z) * y\)

Consider \(0 * y = (x * x) * y = (x * y) * x\)

Now \((x * y) * (0 * y) = (x * y) * ((x * y) * x) = x\)

Hence SG – algebra is a BG – algebra.

**Theorem 3.11**

Every BCK – algebra is a SG– algebra

**Proof**

\(x * (x * y) = y\) is directly follows from Definition of BCK – algebra

\(x * 0 = x * (x * x) = x\)

Therefore BCK – algebra is a SG – algebra.

**Remark 3.12**

The converse is not true as shown in Example 3.3. Then \((X, *, 0)\) is a SG – algebra, but not BCK – algebra, since \((0 * 1) * (0 * 2) \neq 2 * 1\).

**Theorem 3.13**

Every QS – algebra is a SG – algebra

**Proof**

Let \((X, *, 0)\) be a QS – algebra.

Clearly, \(x * 0 = x\) for all \(x \in X\)

To prove \(x * (x * y) = y\)

But we have \((x * y) * (x * z) = z * y\)

Put \(y = 0, (x * 0) * (x * z) = z * 0\)

\(x * (x * z) = z\).

**Remark 3.14**

The converse is not true as shown in Example 3.3. Then \((X, *, 0)\) is a SG – algebra, but not QS – algebra, since \((0 * 1) * (0 * 2) \neq 2 * 1\).

**Theorem 3.15**

Every TM – algebra is a SG – algebra

**Proof**

Let \((X, *, 0)\) be a TM – algebra.

\(x * 0 = x\) is directly follows

To Prove \(x * (x * y) = y\)

Since \((x * y) * (x * z) = z * y\)

Put \(y = 0, (x * 0) * (x * z) = z * 0\)

\(x * (x * z) = z\)

Hence every TM – algebra is a SG – algebra.

**Remark 3.16**

The converse is not true as shown in Example 3.3. Then \((X, *, 0)\) is a SG – algebra, but not TM – algebra, since \((0 * 1) * (0 * 2) \neq 2 * 1\).

**REFERENCES**