ISSN (Online) 2456 -1304

International Journal of Science, Engineering and Management (IJSEM)
Vol 2, Issue 12, December 2017

Some Structures of Idempotent Commutative Semigroup

[1] D. Radha, [2] P. Meenakshi
[1][2] Assistant Professor of Mathematics
[1][2] A. P. C. Mahalaxmi College for Women, Thoothukudi,Tamil Nadu, India.

Abstract:-- **The algebraic theory of semigroups was developed by A. H. Clifford and G. B. Preston [2] and it was extended by several authors like David. McLean [4]. The algebraic theory of commutative semigroup was studied and extended by various authors like M. A. Taiclin [13], A. P. Biryukov [1]. In this paper, we have defined some structures of Idempotent Commutative Semigroup. We have given a notion of left(right) normal, left(right) quasi-normal, regular, normal, left(right) semi-normal, left(right) semi-regular, rectangular, reduced in a Idempotent Commutative Semigroup S. We have proved various theorems like an Idempotent Commutative Semigroup S is left(right) normal if and only if left(right) quasi- normal; S is regular implies normal and vice versa. Further we also proved that S is left(right) semi- normal if and only if left(right) semi-regular; S is left(right) quasi-normal implies left(right) semi-regular and vice versa. We also verified that S is left(right) quasi-normal if and only if left(right) semi-normal. Further it is proved that every left singular with rectangular semigroup is reduced.**

Keywords:-- **Semigroup, Identity, Idempotent elements, Commutativity, rectangular and reduced elements.**

1. INTRODUCTION

The formal study of semigroups began in the early 20th century. Early results include a Cayley theorem for semigroups realizing any semigroup as transformation semigroup, in which arbitrary functions replace the role of bijections from group theory. Other fundamental techniques of studying semigroups like Green's relations do not imitate anything in group theory though. A deep result in the classification of finite semigroups is Krohn–Rhodes theory. The theory of finite semigroups has been of particular importance in theoretical computer science since the 1950s because of the natural link between finite semigroups and finite automata via the syntactic monoid. In probability theory, semigroups are associated with Markov processes. In other areas of applied mathematics, semigroups are fundamental models for linear time-invariant systems.

In mathematics, a semigroup is an algebraic structure consisting of a set together with an associative binary operation. The binary operation of a semigroup is most often denoted multiplicatively: $x \cdot y$, or simply xy, denotes the result of applying the semigroup operation to the ordered pair (x, y). Associativity is formally expressed as that $(x \cdot y) \cdot z = x \cdot (y \cdot z)$ for all x, y and z in the semigroup.

The name "semigroup" originates in the fact that a semigroup generalizes a group by preserving only associativity and closure under the binary operation from the axioms defining a group. From the opposite point of view (of adding rather than removing axioms), a semigroup is an associative magma. As in the case of groups or magmas, the semigroup operation need not be commutative, so $x \cdot y$ is not necessarily equal to $y \cdot x$; a typical example of associative but non-commutative operation is matrix multiplication. If the semigroup operation is commutative, then the semigroup is called a *commutative semigroup* or (less often than in the analogous case of groups) it may be called an *abelian semigroup*.

2.Preliminaries:

In this section we present some basic concepts of semigroups and definitions needed for the study of this chapter.

2.1 Definition

A Semigroup (S, .) is said to be left(right) singular if it satisfies the identity $ab = a(ab = b)$ for all a, b in S.

2.2 Definition

A Semigroup (S, .) is rectangular if it satisfies the identity $aba = a$ for all a, b in S.

2.3 Definition

A Semigroup (S, .) is called left(right) regular if it satisfies the identity $aba = ab(aba = ba)$for all a, b in S.

2.4 Definition

A Semigroup (S, .) is called regular if it satisfies the identity $abca = abaca$ for all a,b,c in S.

2.5 Definition

A Semigroup (S, .) is said to be total if every element of S can be written as the product of two elements of S. i.e, $S^2 = S$.

2.6 Definition

A Semigroup (S, .) is said to be left(right) normal if $abc = acb(abc = bac)$ for all a, b, c in S.

2.7 Definition

A Semigroup (S, .) is said to be normal if it satisfies the identity $abca = acba$ for all a, b, c in S.

2.8 Definition

A Semigroup (S, .) is said to be left(right) quasi-normal if it satisfies the identity $abc = acbc(abc = abac)$ for all a, b, c in S.

2.9 Definition

A Semigroup (S, .) is said to be left(right) semi-normal if it satisfies the identity $abca = acbca(abca = abcba)$ for all a, b, c in S.

2.10 Definition

A Semigroup (S, .) is said to be left(right) semi-regular if it satisfies the identity $abca = abacabca(abca = abcabaca)$ for all a, b, c in S.

3. Structures of Idempotent Commutative Semigroup

In this section, we will see various theorem on Idempotent Commutative Semigroup satisfying some properties.

3.1 Theorem

An Idempotent Commutative Semigroup S is left(right) normal iff it is left(right) quasi-normal.

Proof:

Let (S, .) be an Idempotent Commutative Semigroup.

Now let (S, .) be left normal, then $abc = acb \Longrightarrow abc.c = acbc \Longrightarrow abc = acbc(c.c = c)$

Therefore (S, .) is left quasi-normal.

Conversely. Let (S, .) be left quasi-normal, then $abc = acbc \Longrightarrow abc = accb\ (bc = cb) \Longrightarrow abc = acb(c.c = c)$

Therefore (S, .) is left normal

Now let (S, .) be right normal, then $abc = bac \Longrightarrow aabc = abac \Longrightarrow abc = abac\ (a.a = a)$.

Therefore (S, .) is right quasi normal.

Conversely let (S, .) is right quasi normal, then $abc = abac \Longrightarrow abc = baac\ (a.b = b.a) \Longrightarrow abc = bac(a.a = a)$.

Hence (S, .) is right normal.

3.2 Theorem

An idempotent commutative semigroup S is regular iff it is normal

Proof:

Let (S, .) be an idempotent Commutative Semigroup.

Assume that (S, .) is regular then $abca = abaca \Longrightarrow abca = abcaa\ (a.c = c.a) \Longrightarrow abca = abca \Longrightarrow abca = acba\ (b.c = c.b)$. Therefore (S, .) is normal.

Conversely, assume that (S, .) is normal then, $abca = acba \Longrightarrow abca = abca\ (c.b = b.c) \Longrightarrow abca = a.abca\ (a = a.a)$

$\Longrightarrow abca = abaca\ (a.b = b.a)$

Hence (S, .) is regular.

3.3 Theorem

An Idempotent Commutative Semigroup S is left(right) semi-normal iff it is left(right) semi-regular.

Proof:

Let (S, .) be an idempotent Commutative Semigroup.

Assume that (S, .) is left semi-normal then, $abca = acbca$

$\Longrightarrow abca = aacbbca(a.a = a\ \&\ b.b = b) \Longrightarrow abca = aabcbca\ (b.c = c.b)$

$\Longrightarrow abca = abacbca\ (a.b = b.a)$

$\Longrightarrow abca = abcabca\ (c.a = a.c) \Longrightarrow abca = aabcabca\ (a.a = a)$

$\Longrightarrow abca = abacabca(a.b = b.a)$

Hence (S, .) is left semi-regular.

Conversely, assume that (S, .) is left semi-regular then, $abca = abacabca$

$\Longrightarrow abca = abaacbca\ (a.c = c.a) \Longrightarrow abca = abacbca\ (a.a = a)$

$\Longrightarrow abca = aabcbca\ (a.b = b.a)$

$\Longrightarrow abca = acbbca(a.a = a\ \&\ c.b = b.c) \Longrightarrow abca = acbca\ (b.b = b)$

$\Longrightarrow abca = acbca$. Hence (S, .) is left semi-normal.

Now assume that (S, .) is right semi-normal then, $abca = abcba \Longrightarrow abca = aabccba\ (a.a = a\ \&\ c.c = c) \Longrightarrow abca = abacbca\ (a.b = b.a\ \&\ c.b = b.c) \Longrightarrow abca = abaacbca\ (a.a = a) \Longrightarrow abca = abacabca \Longrightarrow abca = abcabaca\ (a.b = b.a\ \&\ c.a = a.c)$

Hence (S, .) is right semi-regular.

Conversely assume that (S, .) is right semi-regular then, $abca = abcabaca$

$\Longrightarrow abca = abcaabca\ (a.b = b.a) \Longrightarrow abca = abcabca$

$\Longrightarrow abca = abcbaca\ (a.b = b.a) \Longrightarrow abca = abcbcaa\ (c.a = a.c) \Longrightarrow abca = abcbca$

$\Longrightarrow abca = abccba(c.b = b.c) \Longrightarrow abca = abcba\ (c.c = c)$

Hence (S, .) is right semi-normal.

3.4 Theorem

An Idempotent Commutative Semigroup S is left(right) quasi-normal iff it is left(right) semi-regular.

Proof:

Let (S, .) be an idempotent Commutative Semigroup.

Assume that (S, .) is left quasi-normal then, $abc = acbc \Longrightarrow abca = acbca$

$\Longrightarrow abca = aacbbca\ (a.a = a\ \&\ b.b = b) \Longrightarrow abca = aabcbca\ (b.c = c.b)$

$\Longrightarrow abca = abacbca(a.b = b.a) \Longrightarrow abca = abcabca(c.a = a.c)$

$\Longrightarrow abca = aabcabca(a.a = a) \Longrightarrow abca = abacabca\ (b.a = a.b) \Longrightarrow abca = abacabca$

Hence (S, .) is left semi-regular.

Conversely, assume (S, .) is left semi-regular then, $abca = abacabca \Longrightarrow abac = aabcabca(a.c = c.a\ \&\ a.b = b.a)$

$\Longrightarrow aabc = acbabca(a.b = b.a\&\ a.a = a\ \&\ b.c = c.b)$

$\Longrightarrow abc = acabbac\ (a.b = b.a\ \&a.c = c.a) \Longrightarrow abc = acabac\ (b.b = b)$

$\Longrightarrow abc = acaabc\ (a.b = b.a) \Longrightarrow abc = acabc(a.a = a) \Longrightarrow abc = aacbc(a.c = c.a)$

$\Longrightarrow abc = acbc(a.a = a)$. Hence (S, .) is left quasi-normal.

Now assume that (S, .) is right quasi-normal then, $abc = abac \Rightarrow abca = abaca$
$\Rightarrow abca = aabbacca\ (a.a = a, b.b = b\ \& c.c = c)$
$\Rightarrow abca = ababcaca(a.b = b.a\ \& c.a = a.c) \Rightarrow abca = abacbaca(c.b = b.c)$
$\Rightarrow abca = abcabaca\ (a.c = c.a)$ Hence (S, .) is right semi-regular.

Conversely assume that (S, .) is right semi-regular then, $abca = abcabaca$
$\Rightarrow abac = abacbaac(a.c = c.a) \Rightarrow aabc = aabcbac(a.a = a\ \& a.b = b.a)$
$\Rightarrow abc = abcabc(a.a = a\ \& a.b = b.a) \Rightarrow abc = abaccb(a.c = c.a\ \& c.b = b.c)$
$\Rightarrow abc = abacb(c.c = c) \Rightarrow abc = ababc(c.b = b.c) \Rightarrow abc = abac(a.b = b.a\ \& b.b = b)$
Hence (S, .) is right quasi-normal.

3.5 Theorem

An Idempotent Commutative Semigroup (S, .) is left(right) quasi-normal iff left(right) semi-normal.

Proof:

Let (S, .) be an idempotent Commutative Semigroup.
Assume (S, .) is left quasi-normal then, $abc = acbc \Rightarrow abca = acbca$. Hence (S, .) is left semi-normal.
Conversely assume that (S, .) is left semi-normal then, $abca = acbca$
$\Rightarrow abac = acbac(a.c = c.a) \Rightarrow aabc = acabc(a.b = b.a)$
$\Rightarrow abc = acbc(a.a = a\ \& a.c = c.a)$. Hence (S, .) is left quasi- normal.
Now assume that (S, .) is right quasi-normal then, $abc = abac \Rightarrow abca = abaca$
$\Rightarrow abbaca(b.b = b) \Rightarrow abca = ababca(a.b = b.a)$
$\Rightarrow abca = aabcba(a.b = b.a\ \& c.b = b.c) \Rightarrow abca = abcba(a.a = a)$
Hence (S, .) is right semi-normal.
Conversely let (S, .) is right semi-normal then, $abca = abcba$
$\Rightarrow abac = abbca(a.c = c.a\ \& b.c = c.b) \Rightarrow aabc = abac(b.b = b\ \& c.a = a.c)$
$\Rightarrow abc = abac\ (a.a = a)$. Hence (S, .) is right quasi-normal.

3.6 Theorem

An Idempotent Commutative Semigroup (S, .) is left(right) quasi-normal iff it is right(left) semi-normal.

Proof:

Let (S, .) be an idempotent Commutative Semigroup.
Assume that (S, .) is left quasi-normal then $abc = acbc \Rightarrow abca = acbca$
$\Rightarrow abca = abcca(b.c = c.b) \Rightarrow abca = abca(c.c = c) \Rightarrow abca = abbca(b.b = b)$
$\Rightarrow abca = abcba(b.c = c.b)$. Hence (S, .) is right semi-normal.
Conversely assume that (S, .) is right semi-normal then $abca = abcba$
$\Rightarrow abac = acbba(a.c = c.a\ \& b.c = c.b) \Rightarrow aabc = acba(b.b = b\ \& a.b = b.a)$
$\Rightarrow abc = acab(a.a = a\ \& a.b = b.a) \Rightarrow abc = aacb(a.c = c.a) \Rightarrow abc = acb(a.a = a)$
$\Rightarrow abc.c = acb.c \Rightarrow abc = acbc(c.c = c)$. Hence (S, .) is left quasi-normal.
Now assume that (S, .) is right quasi-normal then, $abc = abac \Rightarrow abca = abaca$
$\Rightarrow abca = aabca(a.b = b.a) \Rightarrow abca = abca(a.a = a) \Rightarrow abca = abcca(c.c = c)$
$\Rightarrow abca = acbca(b.c = c.b)$. Therefore (S, .) is left semi-normal.
Conversely let (S, .) be left semi-normal then, $abca = acbca$
$\Rightarrow abac = abcca(a.c = c.a\ \& b.c = c.b) \Rightarrow aabc = abca(a.a = a\ \& a.b = b.a)$
$\Rightarrow abc = abac(a.a = a\ \& a.c = c.a)$
Hence (S, .) is right quasi-normal.

3.7 Theorem

An Idempotent Commutative semigroup (S, .) is left(right) quasi-normal iff it is right(left) semi-regular

Proof:

Let (S, .) be an idempotent Commutative Semigroup.
Assume that (S, .) is left quasi-normal then, $abc = acbc \Rightarrow abca = acbca$
$\Rightarrow abca = aacbbcaa(a.a = a\ \& b.b = b) \Rightarrow abca = aabcbaca(b.c = c.b\ \& a.c = c.a)$
$\Rightarrow abca = abacbaca(a.b = b.a) \Rightarrow abca = abcabaca(a.c = c.a)$. Therefore (S, .) is right semi-regular.
Conversely assume that (S, .) is right semi-regular then, $abca = abcabaca$
$\Rightarrow abac = acbbaaca(a.c = c.a\ \& b.c = c.b\ \& a.b = b.a)$
$\Rightarrow aabc = acbaca(a.b = b.a\ \& b.b = b\ \& a.a = a) \Rightarrow abc = acbaac(a.c = c.a)$
$\Rightarrow abc = acbac(a.a = a) \Rightarrow abc = acabc(a.b = b.a) \Rightarrow abc = aacbc(a.c = c.a)$
$\Rightarrow abc = acbc(a.a = a)$. Therefore (S, .) is left quasi-normal.
Now assume that (S, .) is right quasi-normal then, $abc = abac \Rightarrow abca = abaca$
$\Rightarrow abca = abbaccaa(b.b = b\ \& c.c = c\ \& a.a = a)$
$\Rightarrow abca = ababcaca(a.b = b.a\ \& a.c = c.a) \Rightarrow abca = abacbaca(b.c = c.b)$
$\Rightarrow abca = abacabca(a.b = b.a)$. Therefore (S, .) is left semi-regular.
Conversely, let (S, .) is left semi-regular then, $abca = abacabca$
$\Rightarrow abac = abaacbca(a.c = c.a)$
$\Rightarrow aabc = ababcca\ (a.b = b.a\ \& a.a = a\ \& b.c = c.b) \Rightarrow ab = abbaca(c.c = c\ \& a.b = b.a)$
$\Rightarrow abc = abaac(b.b = b\ \& a.c = c.a)$

$\Longrightarrow$ abc = abac(a. a = a).
Therefore (S, .) is right quasi-normal.

3.8 Theorem

An Idempotent Commutative Semigroup (S, .) is left(right) semi-normal iff it is right(left) semi-regular.

Proof:

Let (S, .) be an idempotent Commutative Semigroup.
Assume that (S, .) is left semi-normal then, abca = acbca $\Longrightarrow$ abca = abcca(b. c = c. b)
$\Longrightarrow$ abca = abcac(a. c = c. a) $\Longrightarrow$ abca = abbcaac(b. b = b & $a.a = a$)
$\Longrightarrow$ abca = abcbaca(b. c = c. b & $c.a = a.c$) $\Longrightarrow$ abca = abcbaaca(a. a = a)
$\Longrightarrow$ abca = abcabaca(a. b = b. a) Therefore (S, .) is right semi-regular.
Conversely assume that (S, .) is right semi-regular then, abca = abcabaca
$\Longrightarrow$ abca = acbbaaca(b. c = c. b & $b.a = a.b$) $\Longrightarrow$ abca = acbaca(b. b = b & $a.a = a$)
$\Longrightarrow$ abca = acbcaa(a. c = c. a) $\Longrightarrow$ abca = acbca(a. a = a).
Therefore (S, .) is left semi-normal.
Now assume that (S, .) is right semi-normal then, abca = abcba
$\Longrightarrow$ abca = aabccbaa(a. a = a & $c.c = c$) $\Longrightarrow$ abca = abacbcaa(a. b = b. a & $b.c = c.b$)
$\Longrightarrow$ abca = abacbaca(a. c = c. a) $\Longrightarrow$ abca = abacabca(a. b = b. a).
Therefore (S, .) is left semi-regular.
Conversely assume that (S, .) is left semi-regular then, abca = abacabca
$\Longrightarrow$ abca = aabcbaca(a. b = b. a) $\Longrightarrow$ abca = abcbcaa(a. a = a & $a.c = c.a$)
$\Longrightarrow$ abca = abcbca(a. a = a) $\Longrightarrow$ abca = abccba(b. c = c. b)
$\Longrightarrow$ abca = abcba(c. c = c). Therefore (S, .) is right semi-normal.

3.9 Theorem

An Idempotent Commutative Semigroup (S, .) is left(right) regular implies it is left(right) normal.

Proof:

Let (S, .) be an idempotent Commutative Semigroup.
Assume that (S, .) is left regular then, aba = ab $\Longrightarrow$ abac = abc
$\Longrightarrow$ aabc = acb(a. b = b. a & $b.c = c.b$) $\Longrightarrow$ abc = acb(a. a = a)
Therefore (S, .) is left normal.
Now let assume that (S, .) is right regular then, aba = ba $\Longrightarrow$ abac = bac
$\Longrightarrow$ aabc = bac(a. b = b. a) $\Longrightarrow$ abc = bac(a. a = a).
Therefore (S, .) is right normal.

3.10Theorem

An Idempotent Commutative Semigroup (S, .) is left(right) regular implies it is right(left) normal.

Proof:

Let (S, .) be an idempotent Commutative Semigroup.
Assume that (S, .) is left regular then, aba = ab $\Longrightarrow$ abac = abc $\Longrightarrow$ aabc = bac(a. b = b. a)
$\Longrightarrow$ abc = bac(a. a = a)
Therefore (S, .) is right normal.
Now assume that (S, .) is right regular then, aba = ba $\Longrightarrow$ abac = bac
$\Longrightarrow$ aabc = abc(a. b = b. a) $\Longrightarrow$ abc = acb(a. a = a & $b.c = c.b$).
Therefore (S, .) is left normal.

References

[1] A. P. Biryukov, "Some algorithmic problems for finitely defined commutative semigroups," Siberian Math.J.8(1967), 384-391.

[2] A. H. Clifford and G. B. Preston : "The algrbraic theory of semigroups" Math.surveys7;vol.I Amer.math.soc 1961.

[3] A. H. Clifford, "Totally ordered commutative semigroup", Bull. Amer. Math.soc.64(1958), 305-316.

[4] David. McLean "Idempotent Semigroups" Amer.math.monthly; 61;110-113.

[5] J. M. Howie "An introduction to semigroup theory" Academic Press (1976).

[6] J. Justin* "Characterization of the Respective Commutative Semigroups" Institute de Recherche d' Informatique et d' Automatique, France communicated by P. M.Cohn Received September 28, 1970.

[7] Miyuki Yamada and Naoki Kimura "Note on idempotent semigroup II" Proc. Japan Acad. 34;110(1958).

[8] Naoki. Kimura "The structure of idempotent semigroup(1) Proc.Japan Acad., 33, (1957) P.642.

[9] Naoki. Kimura "Note on idempotent semigroups I". Proc. Japan Acad.,33.642(1957)

[10] Naoki. Kimura "Note on idempotent semigroups III". Proc. Japan Acad.,34.113(1958)

[11] Naoki. Kimura "Note on idempotent semigroups IV identities of three variables". Proc. Japan Acad.,34,121-123(1958)

[12] P. Sreenivasulu Reddy & Mulugeta Dawud "Structure of Regular Semigroups".

[13] M. A. Taiclin, "Algorithmic problems for commutative semigroups", Soviet Math.Doki.9(1968), 201-204.

[14] M. A. Taiclin, "On the isomporphism problem for commutative semigroup", Math.USSR Sb. 22(1974), 104-128.