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A Gauss-Type Quadrature Approach for Cauchy-Type Oscillatory Singular Integrals

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Abstract— ***This paper focuses on the approximate evaluation of Cauchy-type oscillatory singular integrals of the form***

$$\int_L \frac{e^{iwz}}{z-\tau}\phi(z)dz; w \in \mathbb{R}, |w| > 1;$$

where L is a directed line segment in the complex plane $\mathbb{C}$ joining from $\tau - l$ to $\tau + l$; $\tau \in \mathbb{C}$ is fixed. A Gauss-type quadrature rule is proposed for the approximate evaluation of line integrals $\int_L \phi(z)dz$. Furthermore, a comprehensive scheme is developed for evaluating the Cauchy-type integrals, with error bounds established and validated through numerical experiments on various reference integrals.

Index Terms: analytic function; Cauchy principal value; oscillatory integral; line integral; error bound.

I. LITERATURE REVIEW

The Cauchy-type oscillatory integral

$$\int_L \frac{e^{iwz}}{z-\tau}\phi(z)dz; w \in \mathbb{C}; |w| > 1 \quad (1)$$

where $\phi(z)$ is an analytic function on $\Omega = \{z \in \mathbb{C}: |z| - \tau < \rho = r|l|; r > 1\}$; and L is the line segment joining from the point $\tau - l$ to $\tau + l$ in the complex plane; has been attracted many mathematicians in past and present also. Recently Hota, Saha, Mohanty and Ojha [10] and in the recent past Wang and Xiang [21], Okacha [18] and Capobianco and Crisculo [3] have dealt with its real counterpart. In short Hota, Saha, Mohanty and Ojha [10] have framed a numerical scheme with the help of the quadrature rule meant for the numerical computation of integrals of Cauchy type

$$\int_{-1}^{1} \frac{\phi(x)}{x}dx \quad (2)$$

Hence after they have constructed quasi-exact quadrature for

$$\int_{-1}^{1} \frac{\cos wx}{x}\phi(x)dx; w \in \mathbb{R}; |w| > 1. \quad (3)$$

and achieved accuracy up to appreciate precession. Assuming ϕ as analytic almost everywhere on $\mathbb{C}$, Wang and Xiang [21] have transformed the integral

$$\int_{-1}^{1} \frac{e^{iwx}}{x-\tau}\phi(x)dx; \quad (4)$$

into two integrals on $[0, \infty)$, decay exponentially faster and applied standard Gauss Laguerre quadrature rule for the efficient evaluation of the integral. Okacha [18] used Hermite interpolation in order to evaluate the integral (1) by integrating the integrand using integration by parts. Whereas Capobianco and Crisculo [3] derived interpolatory quadrature rule with terms of orthogonal polynomials with respect to the Jacobi weight for the approximation of integral (1).

In this paper we are mainly concerned with approximate evaluation of integral (1). For this at first, we formulate a numerical scheme to approximate the integral

$$\int_L \frac{g(z)}{z-\tau}dz; \quad (5)$$

where g is analytic in the complex plane $\mathbb{C}$. Later, the proposed scheme with applicable modification is employed for the evaluation of the integral

$$\int_L \frac{e^{iwz}}{z-\tau}\phi(z)dz; w \in \mathbb{R}; |w| > 1.$$

II. FORMULATION OF QUADRATURE SCHEME

Subtracting out the singularity at $z = \tau$ integral (1) reduces to

$$I = \int_{\tau-l}^{\tau+l} e^{iwz}\frac{\phi(z)}{z-\tau}dz$$

$$= \int_{\tau-l}^{\tau+l} (e^{iwz} - e^{iw\tau})\frac{\phi(z)}{z-\tau}dz + e^{iw\tau}\int_{\tau-l}^{\tau+l} \frac{\phi(z)}{z-\tau}dz$$

$$= I_0 + e^{iw\tau}\int_{\tau-l}^{\tau+l} \frac{\phi(z)-\phi(\tau)}{z-\tau}dz + e^{iw\tau}\phi(\tau)\int_{\tau-l}^{\tau+l} \frac{1}{z-\tau}dz$$

$$= I_0 + e^{iw\tau}\int_{\tau-l}^{\tau+l} \frac{\phi(z)-\phi(\tau)}{z-\tau}dz; \quad (6)$$

since the Cauchy Principal value of the rightmost above integral with the transformation

$z = \tau + lt;\ -1 \le t \le 1 (6)$

i.e.

$$\int_{-1}^{1} \frac{dt}{t}$$

$$i\left[\lim_{\epsilon \to 0-} \int_{-1}^{\epsilon} \frac{dt}{t} + \lim_{\epsilon \to 0+} \int_{\epsilon}^{1} \frac{dt}{t}\right] = 0 \quad (7)$$

As a result, we define our integral

$I = I_0 + I_L$; (8)

where

$$I_L = \int_{\tau-l}^{\tau+l} g(z)dz; \quad (8)$$

$g(z) = \frac{\phi(z)-\phi(\tau)}{z-\tau}.$ (9)

Since the singularity at $z = \tau$, has been subtracted out from $g(z)$, so we construct the proposed Gauss-type quadrature rule for the efficient evaluation of the integral (7) as below. Further by transforming to the real like with the transformation $z = \tau + lt$, $-1 \le t \le 1$, our above result reduces to

$$I = e^{iw\tau} \int_{-1}^{1} e^{iht} g(\tau + lt)dt +$$

$$i\phi(\tau) \int_{-1}^{1} \frac{\sin w(\tau+lt)}{t} dt \quad (10)$$

It is seen that the Brikhoof - Young [2] quadrature rule

$$R_{BY}(\phi) = \frac{h}{15}[24\phi(\tau) + 4\{\phi(\tau + l) + \phi(\tau - l)\} - \{\phi(\tau + ih) + \phi(\tau - ih)\}] \quad (11)$$

meant for the approximation of

$$\int_L \phi(z)dz; \quad (12)$$

has been modified by many mathematicians researching this field. Each of these rules is the parametric rule with parameter α, β (in some cases two parametric) where values of these parameters are seen as the fourth root of a real number $x \in (0,1)$. Further, each of these rules uses $z = 0$ as a quadrature node. Keeping these facts in mind, we construct an n-point quadrature formula

$$R_n(\phi) \simeq \int_L g(z)dz \quad (13)$$

where quadrature nodes are the roots of the polynomial

$$P_n(z) = (z - k) \prod_{k=1}^{n} (z^4 - x_k); \quad x_k \in (0,1) \quad (14)$$

i.e. in symbol

$$R_n(g) = \sum_{i=0}^{k-1} w_i g^{(i)}(0)$$

$$+ \sum_{j=1}^{\Upsilon} [\lambda_j\{g(\tau + \alpha_j h) + g(\tau - \alpha_j h)\}$$

$$+ \eta_j\{g(\tau + i\alpha_j h) + g(\tau - \alpha_j h)\}] \quad (15)$$

where $\Upsilon = \lfloor\frac{n}{4}\rfloor$ and $\lambda_k = n - 4\lfloor\frac{n}{4}\rfloor$. To be honest we formulate the rule $R_n(g)$ type for $n = \Upsilon, \Upsilon + 1, \Upsilon + 2, \Upsilon + 3$. In fact, it is found that for $w_1 = 0$ the rule $R_n(g) = I(g)$; for $g(z) = z$. This suffices $R_{\Upsilon+1}(g)$ is identically equal to $R_{\Upsilon+2}(g)$. At this stage, we prove the polynomial defined in equation (10) is orthogonal over L and the existence of such quadrature $R_n(g)$ of the maximum degree of exactness $6\Upsilon + \mu$ where

$$\mu = \begin{cases} \zeta - 1 & for\ \zeta = 0,2 \\ \zeta & for\ \zeta = 1,3. \end{cases}$$

Theorem 2.1 *Suppose the moments $m_k = \int_{-1}^{1} z^k$ exists; for $k \ge 0$,. Then for any $n \in \mathbb{N}$ there exists a unique interpolatory quadrature $R_n(g)$ with a maximul degree of exactness $d_{max} = 6\Upsilon + \mu$, where*

$$\Upsilon = \lfloor\frac{n}{4}\rfloor,\ \lambda_k = n - 4\lfloor\frac{n}{4}\rfloor,\ \mu = \begin{cases} k - 1 & for\ k = 0,2 \\ k & for\ k = 1,3 \end{cases} \quad (16)$$

The node polynomial (2.11) is characterized by the following orthogonality relations

$$\int_0^1 t^k P_n(t^2) t^{\mu/2} w(\sqrt{t})dt = 0 \quad (17)$$

Proof. Let $\phi \in P_d$, where $d \ge n = 4\Upsilon + k$, with $\Upsilon = \lfloor\frac{n}{4}\rfloor$ and $k = n - 4\Upsilon$.

Then, ϕ can be expressed as

$$\phi(z) = u(z)w_n(z) + v(z) = u(z)z^k p_n(z^4) + v(z),\ u \in P_{d-n},\ v \in P_{n_1}, \quad (18)$$

from which, by an integration with respect to the weight function w, we get

$$I(\phi) = \int_{-1}^{1} u(z)z^k p_\Upsilon(z^4)w(z)dz + I(v). \quad (19)$$

Since the quadrature is interpolatory and $v(z) = \phi(z)$ at the zeros of w_n, we have

$$I(v) = Q_n(v) = Q_n(\phi).$$

Thus the quadrature formula $Q_n(\phi)$ has a maximal degree of precision if and only if

$$\int_{-1}^{1} u(z)z^k p_\Upsilon(z^4)w(z)dz = 0. \quad (20)$$

for a maximal degree of the polynomial $u \in P_{d_n}$. According to the values of k, this orthogonality condition can be represented in the form

$$\int_{-1}^{1} h(z^2)z^{\mu+1}p_n(z^4)w(z)dz = 0,\ h \in P_{N-1} \quad (21)$$

which means that the maximal degree of the polynomial $u \in P_{d_n}$ is

$$d_{max} - n = \begin{cases} 2\Upsilon - 1 & for\ k\ is\ even, \\ 2\Upsilon & for\ k\ is\ odd, \end{cases} \quad (22)$$

i.e., $d_{max} = 6\Upsilon + \mu$, where μ is defined by (16).

Finally by substituting $z^2 = t$, the orthogonality condition we get

$$\int_0^1 t^k P_n(t^2) t^{\mu/2} w(\sqrt{t})dt = 0$$

Theorem 2.2 *A unique interpolatory quadrature $R_n(g)$, with a maximum degree of exactness $d_{max} = 6N + \mu$, exists if and only if the polynomial $P_n(t)$ is orthogonal, with respect to the weights $w_j(t) = t^{(v+2j)/3}t^{1/4}$ with $N_j = 1 + [\frac{n-j}{2}]$, j=1, 2*

Proof. From the theorem (2.1), the condition (2.18) may also be written as

$$\int_0^1 z^k p_n(z) z^{\frac{v-1}{4}} w\left(z^{\frac{1}{4}}\right) dz = 0,$$

$$k = 0,1, \dots, n - 1. \quad (23)$$

Now, putting

$$k = 2r + j - 1, r = \lfloor\frac{k}{2}\rfloor$$

we get,

$$\int_0^1 z^r p_n(z) w_j(z) dz = 0,$$

$$r = 0,1, \dots, n_j - 1 \ (j = 1,2) \quad (24)$$

where

$$w_j(z) = z^{\frac{v+2j}{4} - 1} w\left(z^{\frac{1}{4}}\right)\ and\ n_j = 1 + \left\lfloor\frac{n-j}{2}\right\rfloor. \quad (25)$$

Each of the weight functions is defined on $(0, 1)$ and we are getting a relation among these weight functions.

$$w_j(z) = z^{(j-1)/2} w_1(z), \qquad j = 1,2;$$

where

$$w_1(z) = z^{(\nu+2)/4-1.}$$

Again

$$z^{k+(j-1)/2}, \qquad k = 0,1,\dots,n_j - 1; \ j = 1,2;$$

is a Chebyshev system on $(0,\infty)$, and hence on $(0,1)$, and $w_1(z) > 0$ on E. Therefore, $\{w_j,\ j = 1,2\}$ is a complete system, in which all weight functions are supported on the same interval. Hence, orthogonal polynomial $p_n(z)$ has exactly n zeros in (0,1).

To formulate the quadrature rule we calculate the principal part

$\prod_{k=1}^{n} (z^4 - x_k);$ (26)

of our polynomial $P_n(z)$ by

$P_n(x) = \sum_{j=0}^{n} (-1)^j a_j x^{2(n-j)}$ (27)

where

$a_0 = 1$

$a_1 = x_1 + x_2 + x_3 + \dots + x_n$

$a_n = x_1 x_2 \dots x_n$

i.e. in general

$a_j = \sum x_{\theta_1} x_{\theta_2} x_{\theta_3} \dots x_{\theta_k};\ j = 1,2,3,\dots,n;$ (28)

and the summation is taken on all possible combinations of $(\theta_1, \theta_2, \theta_3, \dots, \theta_j)$

Further, the above orthogonality condition reduces to

$$\int_0^1 t^\nu P_n(t^2) t^{\mu/2} dt = 0; \nu = 0,1,2,\dots,(n-1);$$

which directly implies

$$\sum_{j=0}^{n} (-1)^j a_j \int_0^1 t^{\nu+2(n-j)+\frac{\mu}{2}} dt = 0;$$

$\nu = 0,1,2,\dots,(n-1).$ (29)

A. The Proposed Quasi-Exact Method

To construct the method we assume here that the function $\phi(z)$ is continuous and infinitely differentiable in the complex plane $\mathbb{C}$. Now with this assumption expanding $\phi(z)$ by using Taylor's expansion about the singular point $z = \tau$ we get

$$\phi(z) = \sum_{k=0}^{\infty} c_k (z-\tau)^k$$

where $c_k = \frac{\phi^{(k)}(\tau)}{k!}$ are the Taylor's coefficients. Truncating the above series after the first $(n+1)$ terms the interpolating polynomial $g_n(x)$ with the interpolating condition

$$g_n^{(i)}(z) = \phi^{(i)}(z); \forall i = 0(1)n;$$

is obtained as

$$g_n(z) = \phi(\tau) + \sum_{k=1}^{n} \frac{\phi^{(k)}(\tau)}{k!} (z-\tau)^k$$

Applying the standard process it can be shown that the truncation error $\tilde{E}_n(\phi)$ associated with the polynomial $g_n(z)$ is

$$\tilde{E}_n(\phi) = \frac{(z-\tau)^{n+1}}{(n+1)!} \phi^{(n+1)}(\xi);$$

for $\xi \in [\tau - l, \tau + l]$. Now as

$$\phi(z) \simeq g_n(z);$$

thus,

$$J_o(\phi) = \int_{\tau-l}^{\tau+l} (e^{i\omega z} - e^{i\omega\tau}) \frac{\phi(z)}{z-\tau} dz$$

$$\simeq \int_{\tau-l}^{\tau+l} (e^{i\omega z} - e^{i\omega\tau}) \frac{g_n(z)}{z-\tau} dz$$

$$= \phi(\tau) \int_{\tau-l}^{\tau+l} \frac{e^{i\omega z} - e^{i\omega\tau}}{z-\tau} dz$$

$$\sum_{k=1}^{\infty} \frac{\phi^{(k)}(\tau)}{k!} \int_{\tau-l}^{\tau+l} (z-\tau)^{k-1} (e^{i\omega z} - e^{i\omega\tau}) dz$$

$= \phi(\tau)(J_c + iJ_s) + \sum_{k=1}^{n} \frac{\phi^{(k)}(\tau)}{k!} (\delta_{k-1} - \gamma_{k-1});$ (30)

where

$$J_c = \int_{\tau-l}^{\tau+l} \frac{\cos\omega z}{z-\tau} dz = -2 \sin (\omega\tau)\, Si (\omega l),$$

$$J_s = \int_{\tau-l}^{\tau+l} \frac{\sin\omega z}{z-\tau} dz = 2 \cos (\omega\tau)\, Si (\omega l)$$

$$\delta_{k-1} = \int_{\tau-l}^{\tau+l} (z-\tau)^{k-1} (e^{i\omega z}) dz,$$

$$\gamma_{k-1} = \int_{\tau-l}^{\tau+l} e^{i\omega\tau} (z-\tau)^{k-1} dz = \frac{e^{i\omega\tau} l^k}{k} (1 - (-1)^k)$$

Theorem 2.3 *If* $\delta_{k-1} = \int_{\tau-l}^{\tau+l} (z-\tau)^{k-1} (e^{i\omega z}) dz$ and $\delta_i = 0$; for $i \in \mathbb{Z}, i < 0$; then

$i\omega \delta_{k-1} + (k-1)\delta_{k-2} = l^{k-1} e^{i\omega\tau} [e^{i\omega l} - (-1)^{k-1} e^{-i\omega l}];$ (31)

the non-lomogeneous linear recurrence relation holds $\forall k = 1(1)n$.

Proof. Let

$$\delta_{k-1} = \int_{\tau-l}^{\tau+l} (z-\tau)^{k-1} (e^{i\omega z}) dz.$$

By following the method of integration by parts, we have

$$\delta_{k-1} = \left[(z-\tau)^{k-1} \frac{e^{i\omega z}}{i\omega} \right]_{\tau-l}^{\tau+l}$$

$$- \frac{k-1}{i\omega} \int_{\tau-l}^{\tau+l} (z-\tau)^{k-2} (e^{i\omega z}) dz$$

$$= \frac{l^{k-1} e^{i\omega\tau}}{i\omega} [e^{i\omega l} - (-1)^{k-1} e^{-i\omega l}] - \frac{k-1}{i\omega} \delta_{k-2}$$

$$i\omega\delta_{k-1} + (k-1)\delta_{k-2} = l^{k-1} e^{i\omega\tau} [e^{i\omega l} - (-1)^{k-1} e^{-i\omega l}]$$

which complete the prove.

However, the recurrence relation given in equation (31) can be rewritten as

$$\delta_k + B(k-1)\delta_{k-1} = l^k ABC, k = 0,1,2,..$$

where

$$A = e^{i\omega\tau}, B = \frac{1}{i\omega},$$
$$C = e^{i\omega l} - (-1)^{k-1}e^{-i\omega l}$$

On solving by following standard method of solution of recurrence relation we obtain its particular solution as

$$\delta_k = A\sum_{i=1}^{k-1}(-1)^i \frac{k!}{(k-i)!}(e^{i\omega l} - (-1)^{k-i}e^{-i\omega l})\, B^{i+1}l^{k-i} + k!\,(-B)^k\delta_0 \qquad (32)$$

with $\delta_0 = \frac{2}{\omega}e^{i\omega\tau} \quad \sin wh$

B. Error analysis

Let us assume that the function $\phi(x)$ is differentiable a sufficient number of times in $[\tau - l, \tau + l]$. Now with this assumption denoting $E_J(\phi)$ as the error associated with the scheme $S(\phi)$ meant for the numerical integration of the Cauchy type oscillatory integral $J_0(\phi)$ as given in equation (2.32) is obtained as

$$|E_J(\phi)| \le |J_o(\phi) - R_o(\phi)| + e^{i\omega\tau}|I(\phi) - T_n(\phi)|\ |E_o(\phi)| + |E_I(\phi)|; \qquad (33)$$

where

$$E_o(\phi) = J_o(\phi) - R_o(\phi);$$

and

$$E_I(\phi) = e^{i\omega\tau}|I(\phi) - T_n(\phi)|;$$

are the error terms associated with the quadrature rules $R_o(\phi)$ and $T_n(\phi)$ meant for the approximate evaluation of the Filon type oscillatory integral $J_o(\phi)$ and CPV integral $I(\phi)$ respectively. However,

$$|E_o(\phi)| \le \frac{M_{n+1}}{(n+1)!}\left|\int_{\tau-l}^{\tau+l}(z-\tau)^{n+1}(e^{i\omega z} - e^{i\omega\tau})dz\right|$$
$$\le \frac{M_{n+1}}{(n+1)!}\left[\left|\int_{\tau-l}^{\tau+l}(z-\tau)^{n+1}e^{i\omega z}dz\right| + 2\frac{|l|^{n+2}}{n+2}\right] \qquad (34)$$

where $M_{n+1} = \underset{\xi\in L}{Max}|\phi^{(n+1)}(\xi)|$. Further,

$$\left|\int_{\tau-l}^{\tau+l}(z-\tau)^{n+1}e^{i\omega z}dz\right|$$
$$= \frac{1}{|\omega|}\left[2|l|^{n+1} + (n+1)\left|\int_{\tau-l}^{\tau+l}(z-\tau)^n e^{iz}dz\right|\right]$$
$$\le \frac{|l|^{n+1}}{|\omega|}\left[2 + (n+1)\int_{-1}^{1}|t|^n dt\right];$$
$$z = \tau + lt; -1 \le t \le 1;$$
$$= 4\frac{|l|^{n+1}}{|\omega|}$$

As a result,

$$|E_o(\phi)| \le \frac{2|l|^{n+1}M_{n+1}}{(n+1)!}\left[\frac{2}{|\omega|} + \frac{|l|}{n+2}\right].$$

Since, from equation(2.35) it is evident that

$$|E_I(\phi)| \le \frac{2|l|^{11}CM_{11}}{(11)!}; 0 < C < 1;$$

thus,

$$|E_J(\phi)| \le \frac{2|l|^{11}M_{11}}{(11)!}\left[C + \frac{2}{|\omega|} + \frac{|l|}{12}\right]; \qquad (35)$$

for $n = 10$. That is, if we truncate the Taylor's series after the first eleven terms then the scheme will provide at least 10 decimal places of accuracy for an integral of the type(1.1). This fact is vividly seen when the proposed scheme is applied for the evaluation of such types of integrals numerically.

III. NUMERICAL EXPERIMENTS

To test the accuracy of the quadrature rule costructed in the section 2, we have numerically integrated the following integrals by this rule.

$$\mathcal{J}_1 = P\int_{-i}^{i}\frac{e^z}{z}dz,$$
$$\mathcal{J}_2 = P\int_{-i}^{i}\frac{(1+z)e^z}{z}dz$$
$$\mathcal{J}_3 = P\int_{-i}^{i}\frac{(1+zcosz)}{z}dz,$$
$$\mathcal{J}_4 = P\int_{(1-i)/4}^{(-1+i)/4}\frac{tan^{-1}z}{z}dz$$
$$\mathcal{J}_5 = P\int_{(1+i)/2}^{3(1+i)/2}\frac{sinz}{z-(1+i)}dz$$
$$\mathcal{J}_6 = \int_{-i}^{i}e^z dz$$
$$\mathcal{J}_7 = \int_{-i/2}^{i/2}coszdz$$
$$\mathcal{J}_8 = \int_{-(1+i)/\sqrt{2}}^{(1+i)/\sqrt{2}}ze^z dz$$

Table 1: Numerical evaluation of complex Principal value integrals

Integral	Approx. Value of the integral	Absolute Error
$\mathcal{J}_1$	$1.892166140149260i$	5.8×10^{-10}
$\mathcal{J}_2$	$3.575108103339004i$	7.0×10^{-9}
$\mathcal{J}_3$	$2.350402393847402i$	6.6×10^{-9}
$\mathcal{J}_4$	$-0.506613649119302 + 0.492764337999582i$	2.2×10^{-8}
$\mathcal{J}_5$	$1.817558673960816 - 0.205725120904810i$	1.7×10^{-11}

Integral	Approx. Value of the integral	Absolute Error
$\mathcal{J}_6$	$1.682941963189744i$	6.4×10^{-9}
$\mathcal{J}_7$	$1.042190610990673i$	3.2×10^{-12}
$\mathcal{J}_8$	$-0.516830611217106 + 0.422612055568959i$	6.5×10^{-8}

IV. CONCLUSION

This study presents a numerical approach for evaluating Cauchy-type oscillatory integrals. It has been observed that for any value of w, a fixed number of points can achieve consistent accuracy. Consequently, the proposed mixed approach, combining classical and non-classical quadrature techniques, proves effective for the numerical approximation of these integrals.

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