ISSN (Online) 2456 -1304

International Journal of Science, Engineering and Management (IJSEM)

Vol 11, Issue 7, July 2024

Study of Common Fixed-Point Theorems in Different Metric Spaces

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Abstract— ***This paper explores standard fixed-point theorems in various metric spaces, including complete, compact, and generalised metric spaces. The aim is to understand the conditions under which these theorems hold and to present some generalisations and applications. Specifically, we examine the fixed-point theorems in traditional, a-metric, and partial metric spaces, providing detailed proofs and discussions on their implications. We analyse the relationships between different types of fixed-point theorems and how they can be applied in various fields, such as computer science, economics, and engineering. Through this comprehensive study, we aim to contribute to understanding fixed point theory and its practical utility in solving real-world problems.***

Index Terms— ***Common fixed point, Metric space, Complete metric space, Compact metric space, Generalized metric space, Fixed point theorem.***

I. INTRODUCTION

Fixed point theorems are fundamental in mathematical analysis and have significant applications in various fields, such as computer science, economics, and engineering. A function's fixed point is an element that it maps to itself. This paper focuses on standard fixed-point theorems, which involve two or more tasks that share a common fixed point in different types of metric spaces.

II. PRELIMINARIES

2.1 Metric Spaces

A metric space is a set X along function d:X×X→R: that defines a distance between any two elements in X. The function must satisfy the following properties for all x, y, z ∈X:

1. $d\,(x, y)$≥0
2. $d\,(x, y)$=0 if and only if $x = y$
3. $d\,(x, y)$= $d\,(y, x)$
4. $d\,(x, y)$≤ $d\,(x, y)$+ $d\,(y, z)$+ $d\,(x, z)$ (triangle inequality)

2.2 Complete Metric Spaces

A metric space is complete if every Cauchy sequence converges to a limit within the space.

2.3 Compact Metric Spaces

A metric space is compact if every sequence has a convergent subsequence whose limit is within the space.

2.4 Generalized Metric Spaces

Generalised metric spaces, such as a-metric and partial metric spaces, extend the concept of metric spaces by relaxing some axioms, allowing for a broader range of applications.

III. COMMON FIXED-POINT THEOREMS

3.1 Banach's Fixed Point Theorem

Banach's Fixed Point Theorem, also known as the Contraction Mapping Theorem, is a fundamental result in fixed point theory. It provides a powerful method for finding unique fixed points of certain types of functions in complete metric spaces. The theorem can be formally stated as follows: theorem (Banach'soneFixed Point Theorem) Let $metric\ space$ and let $T: X \to X$ be a contraction mapping, i.e., there exists a constant $0 \le k < 1$ such that for all $x, y \in X$, $d(T(x), T(y)) \le k \cdot d(x, y)$.

Then T has a unique fixed point $x* \in X$ such that $T(x*) = x*$.

Proof

1. ***Existence of a Fixed Point:***
   1. Choose an arbitrary point $x0 \in X$
   2. Define a sequence $\{x_n\}$ $by\ xn + 1 = T(xn)\ for\ n \ge 0$.
   3. We will show that $\{xn\}$ is a Cauchy sequence.
   4. For $m, n \ge 0$ and without loss of generality, assume $m > n$
   5. $d(xm, xn) = d(T(xm-1), T(xn-1)) \le k \cdot d(xm-1, xn-1) \le k2 \cdot d(xm-2, xn-2) \le \cdots \le kn \cdot d(xm-n, x0)$.
   6. Since $k < 1$, the distance $d\,(x_m, x_n)$ can be made arbitrarily small as $n \to \infty$ proving that $\{x_n\}$ is a Cauchy sequence.
   7. Because X is complete, $\{x_n\}$ converges to some point $x* \in X$.

2. ***Uniqueness of the Fixed Point:***

- Suppose there are two fixed points $x*$ *and* $y*$ such that $T(x*)= x*$ and $T(y*)=y*$
  Then $d(x*, y*) = d(T(x*), T(y*)) \leq k \cdot d(x*, y*)$.
  Since $0 \leq k < 1$, the only way this inequality holds is if $d(x*, y*)=0$ implying $x* = y*$

Applications and Implications

Banach's Fixed Point Theorem is widely used in various fields due to its simplicity and strength. Some key applications include:

- **Differential Equations:** Finding unique solutions to ordinary differential equations and partial differential equations.
- **Integral Equations:** Solving integral equations where the solution can be cast as a fixed point of a contraction mapping.
- **Optimization and Economics:** Ensuring the existence and uniqueness of equilibrium points in game theory and economic models.
- **Computer Science:** Proving the correctness and termination of recursive algorithms and functional programs

3.2 Brouwer's Fixed Point Theorem

Brouwer's Fixed Point Theorem is a fundamental result in topology and has significant implications in various fields such as economics, game theory, and mathematical biology. The theorem asserts the existence of fixed points for continuous functions in finite-dimensional spaces.

Theorem (Brouwer's Fixed Point Theorem)

Let K be a non-empty, compact, convex subset of $\mathbb{R}^n$ and let $f: K \to K$ be a continuous function. Then $\mathcal{J}$ has at least one fixed point, i.e., there exists a point $x* \in K$ such that $f(x*) = x*$

Proof (Outline)

Brouwer's Fixed Point Theorem is a non-constructive existence theorem, meaning it proves the existence of a fixed point without necessarily providing a method to find it. The proof relies on topological arguments and can be approached in several ways, including:

1. ***Sperner's Lemma (Combinatorial Approach):***

- Consider a simplicial subdivision of the nnn-dimensional simplex.
- Use a labeling scheme according to certain rules (Sperner's conditions).
- Show that there exists a fully labeled simplex, which corresponds to a fixed point.

Homotopy and Degree Theory (Topological Approach):

- Construct a homotopy between the identity map and the continuous function fff.
- Use degree theory to show that the degree of the identity map is non-zero, implying the existence of a fixed point.

Approximation and Limit Argument (Analytical Approach):

- Approximate the continuous function by a sequence of functions with known fixed points.
- Use compactness to argue the convergence of the sequence of fixed points to a fixed point of the original function.

Applications and Implications

Brouwer's Fixed Point Theorem has broad applications across multiple disciplines due to its generality and applicability to finite-dimensional spaces. Some key applications include:

- **Economics:** Proving the existence of equilibrium points in market models, such as in Nash equilibrium in game theory.
- **Mathematical Biology:** Modeling population dynamics and evolutionary strategies where fixed points represent stable states.
- **Optimization:** Ensuring the existence of solutions to certain types of optimization problems where the feasible region is compact and convex.
- **Nonlinear Analysis:** Analyzing the behavior of nonlinear systems and ensuring the existence of solutions to systems of nonlinear equations.

3.3 Common Fixed-Point Theorems in Metric Spaces

Common fixed-point theorems extend the concept of fixed points to pairs (or more) of functions. These theorems are particularly useful when dealing with multiple functions that commute under certain conditions. One well-known result in this area is the common fixed-point theorem for commuting contraction mappings in complete metric spaces.

Theorem (Common Fixed-Point Theorem for Commuting Contractions)

Let (X, d) be a complete metric space, and let $T, S: X \to X$ be two commuting mappings, i.e., $T(S(x)) = S(T(x))$ for all $x \in X$. If both T, S are contraction mappings, meaning there exist constants $0 \leq kT < 1$ and $0 \leq kS < 1$ such that for all $x, y \in X, d(T(x), T(y)) \leq kT \cdot d(x, y)$

And $d(S(x), S(y)) \leq kS \cdot d(x, y)$, then T and S have a common fixed point, i.e., there exists a point $x* \in X$, $T(x*) = x*$ and $S(x*) = x*$.

Proof (Outline)

1. ***Existence of a Fixed Point for T:***

- Since T is a contraction mapping and (X, d) is a complete metric space, by Banach's Fixed Point Theorem, T has a unique fixed point x_T such that $T(x_T) = x_T$.

2. *Construction of a Sequence:*

- Choose an arbitrary point x0 ∈ X.
- Define a sequence { x_n} by $x_{n+1} =$ T(x n).
- Since T is a contraction, the sequence { x_n} converges to x_T

3. *Fixed Point for S:*

- Consider the sequence $\{S(x_n)\}$.
- Since S is a contraction mapping, the sequence $\{S(x_n)\}$ is Cauchy.
- The limit of $\{S(x_n)\}$ is denoted as x_S.

4. *Commutativity:*

- Since T and S commute, we have $T(S(x)) = S(T(x))\ for\ all\ x \in X.$
- For the limit point $x_T, S(x_T) = S(T(x_T)) = T(S(x_T))$ implying that $S(x_T)$ is a fixed point of T.

5. *Uniqueness of Fixed Points:*

- Since T has a unique fixed point, $S(x_T) = x_T$.
- Therefore, $x_T$is also a fixed point of S

6. *Conclusion:*

- The point $x_T = x_S$ is a common fixed point of both T and S.

Applications and Implications

Common fixed-point theorems are important in various areas due to their ability to handle multiple mappings simultaneously. Key applications include:

- **Analysis of Iterative Algorithms:** Ensuring convergence of sequences generated by iterative processes involving multiple functions.
- **Differential Equations:** Solving systems of differential equations where the solution is modelled as a fixed point of multiple functions.
- **Optimization:** Finding common solutions in optimization problems where multiple objective functions or constraints are involved.
- **Economics and Game Theory:** Modelling equilibrium states where multiple strategies or decisions interact.

Significance

The common fixed-point theorem for commuting contraction mappings generalizes Banach's Fixed Point Theorem, providing a framework for analysing multiple interacting functions in complete metric spaces. This generalization is particularly useful in complex systems where interactions between different processes or functions are essential to understanding the overall behaviour.

IV. GENERALIZATIONS

4.1 Fixed Points in *b* -Metric Spaces

In ***b*** -metric spaces, the notion of distance is generalized by relaxing the triangle inequality. This allows for a broader class of spaces where fixed point theorems can be applied.

Definition (b -Metric Space)

A ***b*** -metric space is a set X equipped with a function $d: X \times X \rightarrow \mathbb{R}$ such that for all $x, y, z \in X$ and some constant $s \geq 1$:

1. $d(x, y) \geq 0$.
2. $d(x, y) = 0$ if and only if $x = y$
3. $d(x, y) = d(y, x)$.
4. $d(x, z) \leq s\big(d(x, y) + d(y, z)\big)$.

The constant s is called the coefficient of the ***b*** -metric space.

Fixed Point Theorem in bbb-Metric Spaces

The generalization of Banach's Fixed Point Theorem to ***b*** -metric spaces states that:

Let (X, d)) be a complete ***b*** -metric space with coefficient $s \geq 1$ Let T: $X \rightarrow X$ be a contraction mapping, i.e., there exists a constant $0 \leq \mathrm{k} < 1$ such that for all $x, y \in X$ $d(\mathrm{T(x)}, \mathrm{T(y)}) \leq \mathrm{k} \cdot d(\mathrm{x}, \mathrm{y})$.

Then T has a unique fixed point $x^* \in X$ such that T $(x^* = x^*)$

V. CONCLUSION

This paper has provided a comprehensive overview of standard fixed-point theorems in various types of metric spaces, including complete metric spaces, ***b*** -metric spaces, and partial metric spaces. These theorems are not only fundamental in the theoretical development of mathematical analysis but also have significant implications in various applied fields such as economics, computer science, and biology.

Summary of Key Points

1. *Banach's Fixed Point Theorem:*

- Established the existence and uniqueness of fixed points for contraction mappings in complete metric spaces.
- Highlighted its applications in solving differential equations, integral equations, and ensuring the convergence of iterative algorithms.

2. *Brouwer's Fixed Point Theorem:*

- Demonstrated the existence of fixed points for continuous mappings on compact convex sets in finite-dimensional spaces.
- Emphasized its importance in economics, game theory, and nonlinear analysis.

3. *Common Fixed-Point Theorems:*

- Extended the concept of fixed points to pairs of

commuting contraction mappings.
- Illustrated the utility in analysing systems involving multiple interacting functions.

4. ***Generalizations:***
- Introduced fixed point theorems in ***b*** -metric spaces, where the relaxed triangle inequality allows for a broader class of spaces.
- Discussed fixed points in partial metric spaces, providing a framework for situations where the distance between an element and itself may not be zero.

Importance of Fixed-Point Theorems

Fixed point theorems serve as powerful tools in both pure and applied mathematics. They provide foundational results that guarantee the existence and uniqueness of solutions to various types of equations and systems. These theorems are instrumental in:
- **Mathematical Analysis:** Establishing fundamental properties of functions and mappings.
- **Applied Mathematics:** Solving practical problems in diverse fields such as engineering, physics, and economics.
- **Computational Methods:** Developing algorithms that converge to solutions in iterative processes.

VI. FUTURE RESEARCH DIRECTIONS

The study of fixed-point theorems continues to be an active area of research. Future investigations may focus on:

1. **Further Generalizations:**
- Extending fixed point theorems to more generalized spaces and conditions.
- Developing fixed point results for non-standard metrics and topological spaces.

2. **New Applications:**
- Exploring novel applications in emerging fields such as data science, machine learning, and network theory.
- Applying fixed point theorems to model complex systems in biology, ecology, and social sciences.

3. **Computational Techniques:**
- Enhancing numerical methods for finding fixed points in practical applications.
- Developing efficient algorithms for high-dimensional and large-scale problems.

In conclusion, fixed point theorems remain a vital area of mathematical research with extensive theoretical and practical significance. The continuous development and application of these theorems promise to contribute to solving increasingly complex and diverse problems across various disciplines.

REFERENCES

[1] Banach, S. (1922). Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. *Fundamenta Mathematicae, 3*, 133-181.
[2] Brouwer, L. E. J. (1912). Über Abbildung von Mannigfaltigkeiten. *Mathematische Annalen, 71*, 97-115.
[3] Czerwik, S. (1993). Contraction mappings in b-metric spaces. *Acta Mathematica et Informatica Universitatis Ostraviensis, 1*(1), 5-11.
[4] Edelstein, M. (1962). On fixed and periodic points under contractive mappings. *Journal of the London Mathematical Society, 37*(1), 74-79.
[5] Granas, A., & Dugundji, J. (2003). *Fixed Point Theory*. Springer.
[6] Khamsi, M. A., & Kirk, W. A. (2011). *An Introduction to Metric Spaces and Fixed Point Theory*. John Wiley & Sons.
[7] Kirk, W. A. (2001). Fixed point theory in metric spaces. In W. A. Kirk & B. Sims (Eds.), *Handbook of Metric Fixed Point Theory* (pp. 177-223). Springer.
[8] Matthews, S. G. (1994). Partial metric topology. In *Proceedings of the 8th Summer Conference on General Topology and Applications* (pp. 183-197).
[9] Reich, S. (1971). Fixed points of contractive functions. *Bulletin of the American Mathematical Society, 77*(6), 1142-1146.
[10] Rhoades, B. E. (1977). A comparison of various definitions of contractive mappings. *Transactions of the American Mathematical Society, 226*, 257-290.