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Partition Theory using Generating Function and its Application in Different Fields

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***Abstract*— This paper give ideas about integer partitions and the importance of Euler generating function to learn various properties of partition of integers along with bijective function and Ramanujan congruence related to partition numbers. Applications of the theory of partition of numbers in different fields are briefly discussed.**

***Index Terms:* Partition of number, partition function, generating function, congruences.**

I. INTRODUCTION

Partition theory is a branch of number theory. It is a foundational area of mathematics that connects number theory, combinatorics and algebra. Partition theory gained attention in the 19th century, especially with the work of mathematicians like Ramanujan and G H Hardy. Ramanujan contributed to the development of partition theory by discovering intriguing properties and congruences associated with partition number. The discoveries made by Indian mathematicians in partition theory have influenced modern number theory. Ramanujan's congruences and partition function have led to new insights into modular forms and combinatorial identities, shaping our understanding of mathematical structure.

Definition: If m is a positive integer, then a partition of m is a finite non-increasing sequence of positive integers $\alpha_1, \alpha_2, \ldots, \alpha_r$, such that $\sum_{i=1}^{r} \alpha_i = n$

The α_i's are called parts of partition.

Definition: The partition function p(m) is the number of partition of m i.e. we define partition function p(m) by

$$p(n) = \begin{cases} \text{The number of partition of m} & \text{if } m \epsilon z^+ \\ 1 & \text{if } m = 0 \\ 0 & \text{if m is } -\text{ve int} \end{cases}$$

we characterize partitions as the number of ways in which a given number can

be expressed as a sum of positive integers

For example, ρ(5) = 7 signifies the five different ways we can express the number 5. Therefore, the partitions of the number 5 are:

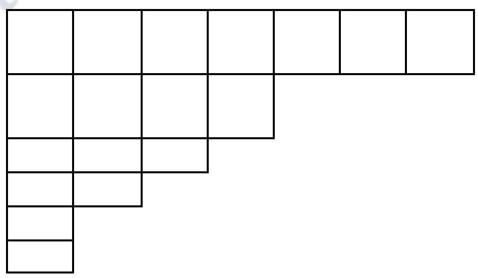
5,
4+1,
3+2,
3+1+1,
2+2+1,
2+1+1+1
1+1+1+1+1.

II. GRAPHICAL REPRESENTATION OF PARTITIONS

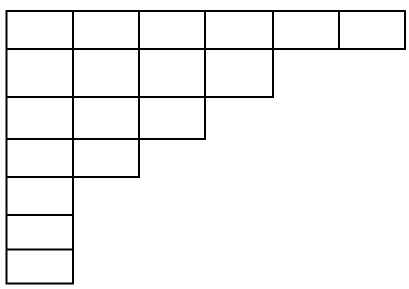
One of the effective elementary method for studying partitions is the graphical representation. The graphical representation or ferrar graph of $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ is an array of dots in which i th row contains α_i dots.

For eg: The graphical representation of 7+4+3+2+1+1 of 18 is

The unit squares can be used instead of dots, so that the graphical representation of 7+4+3+2+1+1 of 18 is



If the above graph is read vertically by columns then this represents the partition as 6+4+3+2+1+1+1



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The above partition is known as the conjugate of the given partition.

Definition: If $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_r)$ is a partition we may define a new partition $\alpha' = (\alpha'_1, \alpha'_2, \ldots, \alpha'_m)$ by choosing α'_i as α that are ≥ I. The partition α' is called the conjugate of α. Equivalently, the graphical representation of the conjugate is obtained by interchanging the rows and columns or reflecting the graph in the main diagonal.

Example: If (5,4,2,1) is the partition of 12.

● ● ● ● ●
● ● ● ●
● ●
●

Then (4,3,2,2,1) is its conjugate

● ● ● ●
● ● ●
● ●
● ●
●

Definition: Self-conjugate partition is one which is identical with its conjugate.

Example: If 3+2+1 is the partition of 6 then its self-conjugate is 3+2+1.

● ● ● ● ● ●
● ● ● ●
● ●

III. GENERATING FUNCTION

Generating function play a crucial role in partition theory as they provide a powerful tool for encoding sequences associated with partitions. Notable Indian mathematicians, including Ramanujan, utilized generating function to derive formulas and explore relationships among partition numbers. By transforming partition problems into algebraic equations, generating function facilitate calculations and several underlying patterns demonstrating their importance in mathematical research on partitions.

Definition: For a given sequence a(n), its generating function is the power series $\sum_{n=0}^{\infty} a(n)x^n$.

For the partition function p(m) the generating function is

$$\sum_{m=0}^{\infty} p(m)x^m = \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots}$$

Theorem: Euler

For $|x| < 1$

$$F_n(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots(1-x^n)}$$

and

$$F(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots} = \sum_{m=0}^{\infty} p(m)x^m$$

Where $F_n(x) = \sum_{m=0}^{\infty} p_n(x)x^m$ is the generating function of $p_n(m)$ - the number of partition of m into parts not exceeding n.

Proof: Suppose that 0<x<1. We have

$$F_n(x) = \frac{1}{(1-x)(1-x^2)(1-x^3)\ldots(1-x^n)}$$

$$= (1 + x + x^2 + \ldots)(1 + x^2 + x^4 + \ldots)\ldots(1 + x^n + x^{2n} + \ldots)$$

$$= (1 + x + x^{1+1} + \ldots)(1 + x^2 + x^{2+2} + \ldots)\ldots(1 + x^n + x^{n+n} + \ldots)$$

These series are absolutely convergent and we can multiply them together and arrange the result as we please. The co-efficient of x^m in the product is $p_n(m)$, the number of partitions of m into parts with each part ≤ n.

Hence $F_n(x) = \sum_{m=0}^{\infty} p_n(m)x^m$

It is clear that

$p_n(m) \le p(m)$ and $p_n(m) = p(m)$, if $m \le n$ and $\lim_{m\to\infty} p_n(m) = p(m)$

Thus,

$$F_n(x) = \sum_{m=0}^{\infty} p_n(m)x^m$$

$$F_n(x) = \sum_{m=0}^{n} p_n(m)x^m + \sum_{m=n+1}^{n} p_n(m)x^m$$

The L.H.S is less than $F(x)$.

i.e., $F_n(x) < F(x)$

$\lim_{n\to\infty} F_n(x) = F(x)$

Thus $\sum_{m=0}^{\infty} p_n(m)x^m < F_n(x) < F(x)$

I.e., $\sum_{m=0}^{n} p(m)x^m < F(x)$

Which implies $\sum_{m=0}^{\infty} p(m)x^m$ converges.

Since $p_n(m) \le p(m)$

$\sum_{m=0}^{\infty} p_n(m)x^m$ converges uniformly in 0 < x < 1.

Thus, $\lim_{n\to\infty} \sum_{m=0}^{\infty} p_n(m)x^m = \sum_{m=0}^{\infty} \lim_{n\to\infty} p_n(m)x^m$

$$= \sum_{m=0}^{\infty} p(m)x^m$$

i.e., $F(x) = \lim_{n\to\infty} F_n(x)$

$= \lim_{n\to\infty} \sum_{m=0}^{\infty} p_n(m)x^m$

$= \sum_{m=0}^{\infty} p(m)x^m$

i.e., $F(x) = \sum_{m=0}^{\infty} p(m)q^m$

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Using generating functions Euler proved many theorems. The following are few theorems:

The number of partitions of m into distinct parts equals the number partitions of m into odd parts.

Let $p_e(D,m)$ denote the number of partitions of m into distinct and even number of parts. Let $p_o(D,m)$ denote the number of partitionns of into distinct and odd number of parts.

Then,

$$p_e(D,m)\text{-}p_o(D,m) = \begin{cases} (-1)^m \ if\ m = \frac{1}{2}n(3n \pm 1) \\ 0\ otherwise \end{cases}$$

Euler proved $p_e(D,m) = p_o(D,m)$ fro all n except those belonging to a special set called pentagonal numbers.

Ramanujan discovered remarkable congruences related to partition number such as

$p(5m + 4) \equiv 0 (\text{mod } 5)$
$p(7m + 5) \equiv 0 (\text{mod } 7)$
$p(11m + 6) \equiv 0 (\text{mod } 11)$

Ramanujan proved the first two congruences using Euler's pentagonal number theorem. Ramanujan gave easy proofs of first and second congruence by writing $\prod_{m=1}^{\infty}(1 - q^m)^4\ and\ \prod_{m=1}^{\infty}(1 - q^m)^6$ as double series using well known identities of Euler and Jacobi.

IV. INFLUENCE OF PARTITION THEORY ON NUMBER THEORY

The partition function p(m), gives a connection between simple integral partitioning and more complex mathematical concepts. Significant impact of the partition function is its role in providing insights into the distribution of prime numbers, modular forms and even in areas such as mathematical and theoretical physics.

A. Applications of partition theory:

While partition theory is primarily a mathematical concept, it has a wide range of applications in various fields. Here are some key areas where partition theory is applied:

1. Combinatorics

- **Counting Partitions:** Partition theory is a important tool for calculating the number of different ways a positive integer can be expressed as a sum of integers, which has applications in combinatorics. For example, counting the number of partitions of an integer is related to finding the number of ways to arrange objects or distribute items into groups.
- **Young Tableaux:** The theory of partitions is deeply connected with young tableaux and symmetric functions. These structures are used in combinatorics to study representations of symmetric groups, which are essential in various areas such as algebraic geometry, coding theory, and more.
- In combinatorics, partition theory can be applied to solve problems related to distributing identical objects into distinct groups. For instance, determining the number of ways to distribute 10 identical balls into 4 distinct boxes, where each box can contain any number of balls, including none. This problem's solution requires understanding the partitions of the number 10 and recognizing how these partitions correspond to the possible distributions of balls into boxes. It exemplifies the practical application of partition theory in solving complex, real-world problems.

2. Algebra

Partition theory also has applications in algebraic structures, such as the theory of symmetric functions, where partitions help in describing the structure Schur functions and in the analysis of representations of symmetric groups.

3. Mathematical physics

- **Statistical Mechanics:** In statistical mechanics, partition functions are used to describe the statistical properties of system in equilibrium. The partition function is a sum over all possible states of the system, weighted by their energies, and is crucial in calculating thermodynamic quantities like free energy, entropyand pressure.
- **Quantum Field Theory (QFT):** Partition functions also appear in QFT, where they represent the sum over all possible field configurations in a quantum system. This connection is important for understanding the behavior of particles and forces at fundamental scales.

4. Cryptography

Partition theory has applications in the development of secure encryption algorithms. In particular, partitions and their generalizations (like compositions) are used in the construction of random keys and cryptographic functions, particularly in systems that require large amounts of randomness and structure.

5. Computer Science

- **Algorithm Design and Complexity:** Partition problems are frequently encountered in algorithms. For example, in combinatorial optimization, the "partition problem" asks if a set can be divided into two subsets with equal sums. This has direct applications in areas such as load balancing, memory allocation, and resource distribution.
- **Data Compression:** Partitioning schemes also find applications in data compression techniques, where data needs to be divided into chunks or segments that are easier to encode or process.

6. Number Theory

- **Ramanujan's Partition Congruences:** The study of partitions is deeply connected to number theory. Famous mathematician Srinivasa Ramanujan discovered certain congruences related to partition numbers. For example, the partition function p(m) gives the number of partitions of an integer m satisfies certain congruences modulo 5, 7, and 11.

- **Modular Forms:** Partition theory plays a extremely important role in the study of modular forms and q-series. The generating function for partitions is a well-known example of a modular form, which has applications in areas like elliptic curves, the theory of modular forms, and the Langlands program.

7. Economics and Game Theory

- **Resource Allocation:** In economics and game theory, partition theory can be used to model the distribution of resources among different agents. For example, a "partition" might represent the different ways in which a set of resources or wealth can be divided among individuals.
- **Shapley Value and Cooperative Games:** The study of partitions is related to cooperative game theory, where the Shapley value is used to allocate resources among players based on their contribution to the overall outcome. The application of partition theory helps understand how groups can be formed and how to share benefits in a fair way.

8. Partitions in Coding Theory

Error Detection and Correction: In coding theory, partition theory can help design codes for error detection and correction. By partitioning a message into different groups of bits, it becomes easier to detect errors or retrieve lost data during transmission.

9. Graph Theory

- **Graph Coloring and Partitioning:** Partition theory has applications in graph theory, particularly in graph coloring and partitioning problems. For example, partitioning the vertices of a graph into disjoint sets with certain properties can be used in algorithms for coloring or finding cliques and independent sets.
- **Graph Isomorphism:** Partitions are also used in understanding and simplifying the structure of graphs. For example, finding a partition of a graph that preserves certain symmetries is essential in graph isomorphism problems.

10. Mathematical Finance

- **Portfolio Theory:** Partition theory is used in portfolio optimization, where the goal is to allocate assets among different categories (e.g., stocks, bonds) to achieve a desired return with a certain level of risk. The problem often involves finding optimal partitions of resources.
- **Risk Management:** Partitions can help model the distribution of financial risks, allowing analysts to segment risks into smaller, manageable parts, each representing a different aspect of risk exposure.

11. Biology

Partition theory is a important tool to understand the dynamics of population studies in biology and epidemiology by modelling the distribution of populations into various states or stages, providing insights into growth patterns and disease spread.

V. CONCLUSION

The basic concept of integer partitions are briefly explained. The idea of generating function is the basic tool to generate partitions of any finite integer. At the end of the paper applications are discussed. Partition theory's applications span a variety of fields, from pure mathematics to theoretical physics, economics, and computer science. Its rich interplay with combinatorics, number theory, and algebra makes it an essential tool in solving a wide range of practical and theoretical problems.

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